Math 210

Quiz # 2, 16 November, 2013

1. (a) State 5 important properties of a continuous function $f : [a, b] \to \mathbb{R}$.

(b) Prove that a continuous function $f : K \to \mathbb{R}$, where K is a compact set, attains its supremum at a point in K.

2. Let $f : X \to Y$ be a given function, and suppose that $f^{-1}(C)$ is an open subset of X whenever C is an open subset of Y.

(a) Prove that f is continuous on X.

(b) Prove that $f^{-1}(B)$ is a closed subset of X whenever B is a closed subset of Y.

(c) If $Y = \mathbb{R}$, and f is continuous, and $a \in \mathbb{R}$, what kind of set is $A = \{x \in X : f(x) \le a\}$? Justify your answer.

3.(a) If f and g are uniformly continuous on a set E, prove that the function $f + g$ is uniformly continuous on $\dot{E}$

(b) Let $h : (0, \infty) \to \mathbb{R}$ be the function defined by

$$h(x) = \cos 2x + x \sin \frac{1}{x}, 0 < x < \infty.$$

Prove that h is uniformly continuous on $(0, \infty)$.

4. A function $f : (a, \infty) \to \mathbb{R}$ is differentiable on (a, ∞), and satisfies

$$\lim_{x\to\infty} (f'(x) + \alpha f(x)) = 0,$$

where α is a positive constant.

(a) Prove that $\lim_{x\to\infty} f(x) = 0$.

(b) Suppose that g is another differentiable function on (a, ∞), that satisfies

$$\lim_{x\to\infty} (g'(x) + \alpha g(x)) = L,$$

where $\alpha > 0$, and L are constants. Find $\lim_{x\to\infty} g(x)$, and $\lim_{x\to\infty} g'(x)$ and prove your answers.

5. Let

$$f(x) = \frac{1 - \cos x}{x}, 0 < x < \pi/2.$$

Prove that f is increasing on $(0, \pi/2)$, and obtain the inequality

$$\sin^2 \frac{x}{2} \le \frac{x}{\pi}, 0 < x < \pi/2$$

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